

# Bouncing Negative-Tension Branes

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## Abstract

Braneworlds, understood here as double domain wall spacetimes, can be described in terms of a linear harmonic function, with kinks at the locations of the boundary branes. In a dynamical setting, there is therefore the risk that the boundary brane of negative tension, at whose location the value of the harmonic function is always lowest, can encounter a zero of this harmonic function, corresponding to the formation of a singularity. We show that for certain types of brane-bound matter this singularity can be avoided, and the negative-tension brane can shield the bulk spacetime from the singularity by bouncing back smoothly before reaching the singularity. In our analysis we compare the 5- and 4-dimensional descriptions of this phenomenon in order to determine the validity of the moduli space approximation.

# 1 Introduction

Recently, a solution to the classical equations of motion of heterotic M-theory was found, which describes a “non-singular” collision of the two boundary branes [1]. By non-singular we mean here that the volume of the internal Calabi-Yau manifold, as well as the scale factors on the branes remain finite and non-zero at the collision, with only the orbifold dimension shrinking to a point. Since domain wall solutions are usually described in terms of a linear harmonic function, one might however expect on general grounds that in a time-dependent context a zero of the harmonic function and thus a spacetime singularity might be encountered at some other point in the evolution. This is indeed the case. The zero of the harmonic function in fact corresponds to a timelike naked singularity, which the negative-tension brane runs into in the absence of matter on the branes. This is the instability described by Gibbons *et al.* [2] and by Chen *et al.* [3]<sup>1</sup>.

However, in the presence of a small amount of certain types of brane-bound matter, the negative-tension brane bounces off the naked singularity *without* touching it. This behaviour is only possible due to the peculiar properties of gravity on a brane of negative tension, and in a sense one can say that in these cases the naked singularity acts repulsively with respect to the negative-tension brane. Thus, and perhaps paradoxically, the negative-tension boundary brane can have a stabilising effect by shielding the bulk spacetime from the naked singularity that corresponds to the zero of the harmonic function (note that because the negative-tension brane corresponds to a trough-like kink, it is always the negative-tension brane, rather than the positive-tension one, which will be the closest to a zero of the harmonic function). It was shown in [5] that from a 4d effective point of view, the bounce of the negative-tension brane corresponds to a reflection of the solution trajectory off a boundary of moduli space. This reflection has the consequence of converting entropy perturbations into curvature perturbations [6], and is thus rather significant in the context of ekpyrotic [7] or cyclic [8] cosmological models. In the present paper we study the conditions for such a bounce to occur in greater generality. What we find is that a certain inequality, involving the trace of the brane matter stress-energy tensor and its coupling to the scalar supporting the domain walls, has to be satisfied in order for a bounce to be possible.

We will study the conditions for a bounce both in 5 dimensions and using the 4d

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<sup>1</sup>It was shown in [4] that static Hořava-Witten braneworlds are stable subject to perturbations of finite energy. However, the time-dependent configurations described in [3] and [1] differ from the static configuration by a homogeneous, infinite-energy perturbation.

moduli space approximation. In the study of higher-dimensional braneworlds, it is often useful to resort to a 4d effective description, since higher-dimensional settings are often quite far removed from one's intuition. It is therefore crucial to determine the validity of the effective theory. We will do this by comparing the description of the bounce of the negative-tension brane from a 5-dimensional point of view with the description of the same phenomenon in the 4-dimensional moduli space approximation, in the presence of various types of brane-bound matter.

## 2 Domain Walls in 5 Dimensions

We will consider scalar-gravity theories with an exponential scalar potential. The action is given by

$$\begin{aligned}
S = & \int_{5d} \sqrt{-g} \left[ R - \frac{1}{2} (\partial\phi)^2 - 6\alpha^2 (3\beta^2 - 2) e^{2\beta\phi} \right] \\
& + 12\alpha \int_{4d, y=-1} \sqrt{-g} e^{\beta\phi} - 12\alpha \int_{4d, y=+1} \sqrt{-g} e^{\beta\phi}, \tag{2.1}
\end{aligned}$$

where  $\alpha$  is a positive constant that can be adjusted by a shift in the scalar  $\phi$  (we will choose a convenient value later on) and  $\beta$  determines the self-coupling of  $\phi$ . Theories of this type are well-motivated in a supergravity context, where they can arise after flux compactification *à la* Scherk-Schwarz, see for example [9]. Typically, the domain wall action is given by a worldvolume-weighted superpotential

$$\mp \int_{4d, y=\pm 1} \sqrt{-g} W(\phi), \tag{2.2}$$

where here  $W(\phi) = 12\alpha e^{\beta\phi}$ . This superpotential is then related to the potential  $V(\phi) = 6\alpha^2 (3\beta^2 - 2) e^{2\beta\phi}$  by the usual supergravity relationship

$$V = \frac{1}{8} \left[ \left( \frac{\partial W}{\partial \phi} \right)^2 - \frac{2}{3} W^2 \right], \tag{2.3}$$

see [10] and the appendix of [11] for more details. The case  $\beta = -1$  corresponds to heterotic M-theory in its simplest consistent truncation [12–15];  $e^\phi$  then parameterises the volume of the internal Calabi-Yau manifold.

The static vacuum of the theory above is given by a domain wall spacetime of the

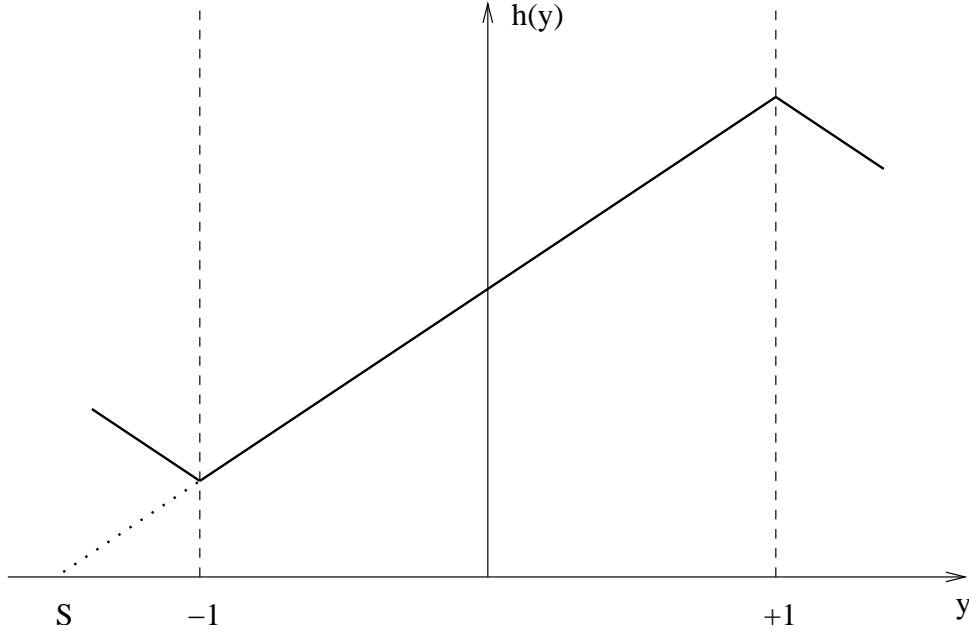


Figure 1: The harmonic function  $h(y)$ , where  $y$  is the coordinate on a  $S^1/\mathbb{Z}_2$  orbifold. In the absence of a negative-tension brane at  $y = -1$ , there would have been a singularity at  $y = S$ .

form

$$\begin{aligned}
ds^2 &= h^{2/(6\beta^2-1)}(y) [B^2 (-d\tau^2 + d\vec{x}^2) + A^2 dy^2], \\
e^\phi &= A^{-1/\beta} h^{-6\beta/(6\beta^2-1)}(y), \\
h(y) &= \alpha (6\beta^2 - 1)y + D,
\end{aligned} \tag{2.4}$$

where  $A$ ,  $B$  and  $D$  are arbitrary constants and  $h(y)$  is a linear harmonic function. The  $y$  coordinate is taken to span the orbifold  $S^1/\mathbb{Z}_2$  with fixed points at  $y = \pm 1$ . In the ‘upstairs’ picture of the solution, obtained by  $\mathbb{Z}_2$ -reflecting the solution across the branes, there is a downward-pointing kink at  $y = -1$  and an upward-pointing kink at  $y = +1$ . These ensure the junction conditions are satisfied, with the negative-tension brane being located at  $y = -1$  and the positive-tension brane at  $y = +1$ . The coordinate system used above is only a good coordinate system when

$$\beta^2 > \frac{1}{6}, \tag{2.5}$$

and we will restrict our analysis to this range of  $\beta$  (as discussed recently in [16], for certain physical properties there are qualitative differences when  $0 \leq \beta^2 \leq \frac{1}{6}$ ).

The Ricci scalar is proportional to  $h^{-12\beta^2/(6\beta^2-1)}$  and thus the spacetime is singular at  $h(y) = 0$ . If we had only a positive-tension brane, with a roof-type kink, this singularity

would be at a finite proper distance from the brane, and the spacetime would therefore have a naked singularity. Usually, one avoids this problem by cutting the spacetime off with a negative-tension brane placed in between the positive-tension brane and the singularity, thereby rendering the spacetime well-behaved, as we have already anticipated by including two brane actions of opposite tension in the action (2.1), see also Figure 1. In a time-dependent context however, where the slope and the height of the harmonic function can vary, there is still the risk that the harmonic function can become zero at the location of the negative-tension brane, thus causing a spacetime singularity to form [2, 3]. In the next section we will see that, in the presence of certain types of brane-bound matter, this singularity can be avoided, with the negative-tension brane bouncing back before it reaches the singularity.

### 3 General Conditions for a Bounce of the Negative-Tension Brane

In general, we add the following matter action at the location of the negative-tension brane (at  $y = -1$ ), *i.e.* we add to equation (2.1) the term:

$$+ \int_{4d, y=-1} \mathcal{L}(g, \phi, \dots), \quad (3.1)$$

where the dots represent the matter contribution and we are allowing for a coupling to the scalar  $\phi$ . The junction conditions, which we are only writing out here for the negative-tension brane, read (in this section  $' \equiv \frac{\partial}{\partial y}$  and  $\cdot \equiv \frac{\partial}{\partial t}$ )

$$a' = \alpha e^{n+\beta\phi} + \frac{1}{6} e^n \mathcal{T}_0 \Big|_{y=-1} \quad (3.2)$$

$$n' = \alpha e^{n+\beta\phi} - \frac{1}{3} e^n \mathcal{T}_0 + \frac{1}{6} e^n \mathcal{T}_i^i \Big|_{y=-1} \quad (3.3)$$

$$\phi' = -6\alpha\beta e^{n+\beta\phi} + \frac{1}{2} e^n \mathcal{T}_\phi \Big|_{y=-1}, \quad (3.4)$$

where we have defined

$$\mathcal{T}_{\mu\nu} \equiv \frac{-1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \quad (3.5)$$

$$\mathcal{T}_\phi \equiv \frac{-1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta \phi}, \quad (3.6)$$

with  $\mu$  a brane worldvolume index. Since the brane as well as the brane-bound matter are kept at the fixed coordinate position  $y = -1$ , we have  $\mathcal{T}_{\mu y} = 0$ .

We are only interested in whether or not the negative-tension brane will bounce off the singularity, even if the bulk is perturbed in the vicinity of this bounce. Therefore we will choose a general metric and scalar field ansatz, which however respects cosmological symmetry on the brane worldvolumes, so that, on the branes, we have spatial homogeneity and isotropy:

$$ds^2 = e^{2n(t,y)}(-dt^2 + dy^2) + e^{2a(t,y)}d\vec{x}^2 \quad (3.7)$$

$$e^\phi = e^{\phi(t,y)}. \quad (3.8)$$

With this metric ansatz we need  $\mathcal{T}_{0i} = 0$  for the  $0i$  Einstein equation to be satisfied.

As a minimal requirement for a bounce to occur, there should be a solution in which the negative-tension brane is momentarily stationary (*i.e.* for which all first time derivatives are zero at the location of the negative-tension brane), and in which the second time derivative of the scale factor on the negative-tension brane is positive. The  $yy$  bulk Einstein equation, which is an equation for the acceleration of the scale factor  $a$ , is given by

$$3\ddot{a} - 3\dot{a}\dot{n} + \frac{1}{4}\dot{\phi}^2 + 6\dot{a}^2 = 3a'^2 + 3a'n' - \frac{1}{4}\phi'^2 + 9\alpha^2(6\beta^2 - 1)e^{2n+2\beta\phi}. \quad (3.9)$$

We can set first time derivatives to zero, since we are only interested here in the moment of the bounce. Apart from the  $ty$  Einstein equation (which is trivially satisfied at  $y = -1$  since every term involves a first time derivative), this equation is the only one that involves only first derivatives with respect to  $y$ , and so we can evaluate it at the location of the negative-tension brane at the moment of the putative bounce by substituting in the junction conditions (3.2)-(3.4):

$$\begin{aligned} 3\ddot{a} = & \frac{\alpha}{2}e^{2n+\beta\phi}(\mathcal{T}_\mu^\mu + 3\beta\mathcal{T}_\phi) \\ & - \frac{e^{2n}}{48}[4(\mathcal{T}_0^0)^2 - 4\mathcal{T}_0^0\mathcal{T}_i^i + 3(\mathcal{T}_\phi)^2] \quad |_{y=-1;\text{bounce}} \end{aligned} \quad (3.10)$$

The first line is proportional to  $\alpha$ , and would therefore flip sign on the positive-tension brane (where there would be additional first time-derivative terms involved). The first line also involves the trace of the matter stress-energy tensor. The second line is proportional to the matter density squared, and can thus be regarded as small compared to the first line. The second line generally gives a negative contribution (it certainly does so when the strong energy condition is satisfied).

If we want to have a bounce on the negative-tension brane, there must be a positive contribution to  $\ddot{a}$  from the first line in (3.10), *i.e.* a necessary condition (but not sufficient

in general) is that

$$\mathcal{T}_\mu^\mu + 3\beta\mathcal{T}_\phi > 0. \quad (3.11)$$

This condition is not particularly difficult to satisfy; we will give a few examples (and counter-examples) in the next section. If equation (3.11) is satisfied, then one also has to check that this contribution is dominant over the second line in (3.10), which it is if the matter density is sufficiently small. And one would of course have to extend the solution to the rest of spacetime, which we simply assume here to be feasible.

## 4 Some Examples

### *Scalar Field*

Using the above equations, one can see that for a scalar matter Lagrangian

$$\mathcal{L} = -\sqrt{-g}\frac{1}{2}(\partial\sigma)^2 C(\phi), \quad (4.1)$$

where we allow for a coupling  $C(\phi)$  and where we take  $\sigma$  to depend only on time (because of the assumed cosmological symmetry), we get a positive contribution to (3.10) when

$$C - 3\beta C_{,\phi} > 0. \quad (4.2)$$

Thus for a scalar field that doesn't couple to  $\phi$ , *i.e.* for which  $C = 1$ , we can expect a bounce; however there will also be corrections to the geometry. Scalars of this latter type are present in heterotic M-theory [15]. We will discuss the heterotic M-theory examples in more detail in section 6.

### *Gauge Field*

A vector gauge field localised on the brane is represented by the Lagrangian

$$\mathcal{L} = -\sqrt{-g}C(\phi)F_{\mu\nu}F^{\mu\nu}. \quad (4.3)$$

Here we assume the gauge field to be abelian, and we use the usual electric-magnetic decomposition

$$F^{0i} = E_i \quad F^{ij} = \epsilon^{ijk}B_k. \quad (4.4)$$

This leads to a stress-energy tensor

$$\mathcal{T}_{00} = -g_{00}(E^2 + B^2)C(\phi) \quad (4.5)$$

$$\mathcal{T}_{0i} = -2\epsilon_{ijk}E^jB^kC(\phi) \quad (4.6)$$

$$\mathcal{T}_{ij} = [-2E_iE_j - 2B_iB_j + g_{ij}(E^2 + B^2)]C(\phi), \quad (4.7)$$

where we have denoted  $B = (B_i B^i)^{1/2}$ . We can immediately see that the stress-energy tensor is traceless,

$$\mathcal{T}_\mu^\mu = 0. \quad (4.8)$$

We also have

$$\mathcal{T}_\phi = C_{,\phi}(-2E^2 + 2B^2). \quad (4.9)$$

The  $oi$  Einstein equation implies that  $\mathcal{T}_{0i}$ , and thus the Poynting vector, has to be zero. This will be the case if we have an electric or a magnetic field only. Thus, from (3.11), we can expect a bounce if

$$\beta C_{,\phi} < 0 \quad \text{and} \quad B_i = 0 \quad (4.10)$$

or if

$$\beta C_{,\phi} > 0 \quad \text{and} \quad E_i = 0. \quad (4.11)$$

On the other hand, it is easy to see that radiation alone, for which the Poynting vector is zero on average, does not give rise to a bounce, since then <sup>2</sup>

$$\langle E^2 \rangle = \langle B^2 \rangle. \quad (4.12)$$

In that case the condition (3.11) cannot be fulfilled, as we now have  $\mathcal{T}_\mu^\mu + 3\beta\mathcal{T}_\phi = 0$ . However, radiation also doesn't lead to a collapse; to first order in the matter density it simply has no effect at all on whether we have a bounce or not. It is only at second order in the energy density that radiation contributes towards a collapse, as can be seen from equation (3.10).

### *Perfect Fluid and Cosmological Constant*

A perfect fluid with energy density  $\rho$  can be described by the Lagrangian

$$\mathcal{L} = -\sqrt{-g}\rho C(\phi), \quad (4.13)$$

which leads to the stress-energy tensor [17]

$$\mathcal{T}_{00} = -g_{00}\frac{1}{2}\rho C(\phi) \quad (4.14)$$

$$\mathcal{T}_{ij} = g_{ij}\frac{1}{2}pC(\phi) \quad (4.15)$$

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<sup>2</sup>In order to perform the averaging, we are assuming here that  $C_{,\phi}$  varies slowly.



and

$$\mathcal{T}_\phi = \rho C_{,\phi}, \quad (4.16)$$

where  $p$  denotes the fluid's pressure. With an equation of state  $p = w\rho$  and  $\rho > 0$ , we get a bounce if

$$\beta C_{,\phi} > \frac{1-3w}{6}C. \quad (4.17)$$

Note that due to the coupling to the scalar  $\phi$ , radiation should not be represented as a perfect fluid with  $w = \frac{1}{3}$ , but rather as a gauge field, as above. In fact, for that same reason, it is doubtful to what extent the perfect fluid effective description is accurate in general, except in the case of a cosmological constant, which we write out explicitly here.

For a brane-localised cosmological constant  $\Lambda$ , we would consider

$$\mathcal{L} = -\sqrt{-g}2\Lambda C(\phi). \quad (4.18)$$

Then

$$\mathcal{T}_{\mu\nu} = -\Lambda g_{\mu\nu}C \quad (4.19)$$

and the condition (3.11) is satisfied for

$$\Lambda(\beta C_{,\phi} - \frac{2}{3}C) > 0. \quad (4.20)$$

Thus, for a positive cosmological constant  $\Lambda > 0$  we can expect a bounce if the coupling is

$$e^{c\phi} \quad \text{with} \quad \beta c > \frac{2}{3}. \quad (4.21)$$

If we have a negative cosmological constant, we can have a bounce if the coupling is

$$e^{c\phi} \quad \text{with} \quad \beta c < \frac{2}{3}. \quad (4.22)$$

Note that when  $C = e^{\beta\phi}$ , the addition of a cosmological constant corresponds to a de-tuning of the brane tensions, since it effectively changes the value of  $\alpha$  in the brane action at  $y = -1$  in equation 2.1.

## 5 The Moduli Space Description

For many reasons, not least because of our lack of intuition about higher-dimensional settings and in order to make contact with what we can observe at present, it is useful to have a 4-dimensional effective description of higher-dimensional physics. An obvious question

however is how much of the higher-dimensional dynamics a 4d effective description can capture. We will address this question by looking at the 4d moduli space approximation for the examples presented in the previous section. The derivation of the moduli space action in this section will be a generalisation to arbitrary  $\beta$  of the derivation in [5], where it was performed for the case  $\beta = -1$ .

To implement the moduli space approximation, we simply promote the moduli of the static solution (2.4) to arbitrary functions of the brane conformal time  $\tau$ , yielding the ansatz:<sup>3</sup>

$$\begin{aligned} ds^2 &= h^{2/(6\beta^2-1)}(\tau, y) [B^2(\tau) (-d\tau^2 + d\vec{x}^2) + A^2(\tau) dy^2], \\ e^\phi &= A^{-1/\beta}(\tau) h^{-6\beta/(6\beta^2-1)}(\tau, y), \\ h(\tau, y) &= \alpha(6\beta^2 - 1)y + D(\tau), \quad -1 \leq y \leq +1. \end{aligned} \quad (5.1)$$

This ansatz satisfies the  $\tau y$  Einstein equation identically, which is important, since otherwise the  $\tau y$  equation would act as a constraint [18]. Having defined the time-dependent moduli, we would now like to derive the action summarising their equations of motion. This is achieved by simply plugging the ansatz (5.1) into the original action (2.1), yielding the result (where we use the notation  $\dot{\phantom{x}} \equiv \partial/\partial\tau$ )

$$\begin{aligned} S_{\text{mod}} &= 6 \int_{4d} AB^2 I_{\frac{3}{6\beta^2-1}} \left[ \frac{1}{12\beta^2} \left( \frac{\dot{A}}{A} \right)^2 - \left( \frac{\dot{B}}{B} \right)^2 - \frac{\dot{A}\dot{B}}{AB} \right. \\ &\quad \left. + \frac{3\beta^2 - 2}{(6\beta^2 - 1)^2} \frac{I_{\frac{-12\beta^2+5}{6\beta^2-1}}}{I_{\frac{3}{6\beta^2-1}}} \dot{D}^2 - \frac{3}{6\beta^2 - 1} \frac{I_{\frac{-6\beta^2+4}{6\beta^2-1}} \dot{B}\dot{D}}{I_{\frac{3}{6\beta^2-1}} B} \right], \end{aligned} \quad (5.2)$$

where we have defined

$$I_n = \int_{-1}^1 dy h^n = \frac{1}{(n+1)\alpha(6\beta^2-1)} [(D + \alpha(6\beta^2-1))^{(n+1)} - (D - \alpha(6\beta^2-1))^{(n+1)}]. \quad (5.3)$$

This action can be greatly simplified by introducing the field redefinitions

$$a_4^2 \equiv AB^2 I_{\frac{3}{6\beta^2-1}}, \quad (5.4)$$

$$e^{\sqrt{\frac{12\beta^2}{3\beta^2+1}}\psi} \equiv A (I_{\frac{3}{6\beta^2-1}})^{3\beta^2/(3\beta^2+1)}, \quad (5.5)$$

$$(6\beta^2 - 1)\chi \equiv - \int dD \frac{[(3\beta^2 - 2) I_{\frac{-12\beta^2+5}{6\beta^2-1}} I_{\frac{3}{6\beta^2-1}} + \frac{9}{12\beta^2+4} (I_{\frac{-6\beta^2+4}{6\beta^2-1}})^2]^{1/2}}{I_{\frac{3}{6\beta^2-1}}}. \quad (5.6)$$

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<sup>3</sup>Note that the relationship between the coordinates  $(\tau, x, y)$  used in this section and the coordinates  $(t, x, y)$  used in the previous section is in general rather complicated. We will not need the corresponding coordinate transformations in this paper.

Note that  $a_4$  has the interpretation of being roughly the four-dimensional scale factor, whereas  $\psi$  and  $\chi$  are four-dimensional scalars. The definition (5.6) can be rewritten as stating that

$$\sqrt{3\beta^2 + 1} d\chi = \frac{-dD}{(D + \alpha(6\beta^2 - 1))^{(3\beta^2 - 2)/(6\beta^2 - 1)} (D - \alpha(6\beta^2 - 1))^{(3\beta^2 - 2)/(6\beta^2 - 1)} I_{\frac{3}{6\beta^2 - 1}}}. \quad (5.7)$$

This expression can be integrated to yield

$$D = \alpha(6\beta^2 - 1) \left[ \frac{(1 + e^{2\sqrt{3\beta^2 + 1}\chi})^{(6\beta^2 - 1)/(3\beta^2 + 1)} + (1 - e^{2\sqrt{3\beta^2 + 1}\chi})^{(6\beta^2 - 1)/(3\beta^2 + 1)}}{(1 + e^{2\sqrt{3\beta^2 + 1}\chi})^{(6\beta^2 - 1)/(3\beta^2 + 1)} - (1 - e^{2\sqrt{3\beta^2 + 1}\chi})^{(6\beta^2 - 1)/(3\beta^2 + 1)}} \right]. \quad (5.8)$$

In terms of  $a_4$ ,  $\psi$  and  $\chi$  the moduli space action (5.2) then reduces to the remarkably simple form

$$\frac{1}{6} S_{\text{mod}} = \int_{4d} [-\dot{a}_4^2 + a_4^2(\dot{\psi}^2 + \dot{\chi}^2)]. \quad (5.9)$$

The minus sign in front of the kinetic term for  $a_4$  is characteristic of gravity, and in fact this is the action for gravity with scale factor  $a_4$  and two minimally coupled scalar fields. Note that all the different 5d theories, with different  $\beta$ , are thus described by the same 4d effective theory to a first approximation. We will see shortly however that the inclusion of brane-bound matter lifts this degeneracy.

Useful expressions relating 4d and 5d quantities at the location of the negative-tension brane are given by:

$$b_- = (\alpha(6\beta^2 + 2))^{1/(6\beta^2 + 2)} a_4 e^{-\sqrt{\frac{3\beta^2}{3\beta^2 + 1}}\psi} (-\sinh \sqrt{3\beta^2 + 1}\chi)^{1/(3\beta^2 + 1)} \quad (5.10)$$

$$e_-^\phi = (\alpha(6\beta^2 + 2))^{-6\beta/(6\beta^2 + 2)} e^{-\frac{2}{\beta}\sqrt{\frac{3\beta^2}{3\beta^2 + 1}}\psi} (-\sinh \sqrt{3\beta^2 + 1}\chi)^{-6\beta/(3\beta^2 + 1)}, \quad (5.11)$$

where  $b_-$  denotes the brane scale factor  $b_- = h^{1/(6\beta^2 - 1)}(\tau, y = -1)B(\tau)$ . Note that since  $b_-$  is a positive quantity, the range of  $\chi$  should be restricted to  $(-\infty, 0]$ . For simplicity we will set  $\alpha = 1/(6\beta^2 + 2)$  in what follows; this can be done by a shift in  $\phi$ . Also, in this section we always assume the coupling function  $C(\phi)$  to be of the form

$$C(\phi) = e^{c\phi}. \quad (5.12)$$

In heterotic M-theory ( $\beta = -1$ ), where the volume of the Calabi-Yau manifold is given by  $e^\phi$ , this corresponds to the brane-bound matter fields coupling to a power of the volume of the internal manifold.

Before continuing, let us present a brief argument which partially explains the simplicity of the moduli space action (5.9). This argument rests on the observation that the original 5d action (2.1) is invariant under the global scaling symmetry

$$g_{mn} \rightarrow e^{2\epsilon} g_{mn} \quad (5.13)$$

$$\phi \rightarrow \phi - \frac{1}{\beta}\epsilon, \quad (5.14)$$

where  $\epsilon$  is a constant parameter. Under this symmetry, the moduli of the domain wall solution (5.1) transform as

$$A \rightarrow e^\epsilon A \quad (5.15)$$

$$B \rightarrow e^\epsilon B \quad (5.16)$$

$$D \rightarrow D. \quad (5.17)$$

This in turn corresponds to the transformations

$$a_4 \rightarrow e^{3\epsilon/2} a_4 \quad (5.18)$$

$$\psi \rightarrow \psi + \sqrt{\frac{3\beta^2 + 1}{12\beta^2}} \epsilon \quad (5.19)$$

$$\chi \rightarrow \chi. \quad (5.20)$$

Thus we see that this symmetry induces the shift symmetry in  $\psi$ . It is also interesting to note that the absence of an implied shift symmetry in  $\chi$  is consistent with the fact that the range of  $\chi$  is actually limited, as noted above, and that the absolute value of  $\chi$  is a meaningful quantity.

### *Scalar Field*

For a scalar field  $\sigma$  coupling to the scalar  $\phi$  via  $e^{c\phi}$ , with  $c$  an arbitrary real number, we get an addition to the effective theory (5.9) of

$$-\sqrt{-g} e^{c\phi} g^{00} \dot{\sigma}^2 \Big|_{y=-1} \quad (5.21)$$

$$= a_4^2 e^{-2(c/\beta+1)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(-6\beta c+2)/(3\beta^2+1)} \dot{\sigma}^2. \quad (5.22)$$

The equation of motion for  $\sigma$  can be solved immediately to give

$$\dot{\sigma} = \frac{\sigma_0}{a_4^2} e^{2(c/\beta+1)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(6\beta c-2)/(3\beta^2+1)}, \quad (5.23)$$

where  $\sigma_0$  is a constant. Also, the equation of motion

$$\frac{\ddot{a}_4}{a_4} = -\dot{\psi}^2 - \dot{\chi}^2 - \frac{\sigma_0^2}{a_4^4} e^{2(c/\beta+1)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(6\beta c-2)/(3\beta^2+1)} \quad (5.24)$$

together with the constraint<sup>4</sup> (Friedmann equation)

$$\frac{\dot{a}_4^2}{a_4^2} = \dot{\psi}^2 + \dot{\chi}^2 + \frac{\sigma_0^2}{a_4^4} e^{2(c/\beta+1)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(6\beta c-2)/(3\beta^2+1)} \quad (5.25)$$

lead to

$$a_4 = \tau^{1/2}. \quad (5.26)$$

If we then define a new time variable

$$T \equiv \ln \tau, \quad (5.27)$$

the remaining equations of motion can be expressed as

$$\psi_{,TT} + \frac{\sigma_0^2}{2} V_{,\psi} = 0 \quad (5.28)$$

$$\chi_{,TT} + \frac{\sigma_0^2}{2} V_{,\chi} = 0, \quad (5.29)$$

or, equivalently, by the action

$$\int_{4d} \psi_{,T}^2 + \chi_{,T}^2 - \sigma_0^2 V(\psi, \chi). \quad (5.30)$$

The effective potential is given by

$$V = e^{2(c/\beta+1)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(6\beta c-2)/(3\beta^2+1)}. \quad (5.31)$$

Therefore, as  $\chi \rightarrow 0$  the effective potential blows up and becomes repulsive if

$$\beta c < 1/3. \quad (5.32)$$

Thus the solution trajectory effectively gets reflected off the  $\chi = 0$  plane which means that the scale factor on the negative-tension brane starts increasing again (see equation (5.10)), *i.e.* the negative-tension brane bounces. Condition (5.32) is the same as that derived above from the 5d point of view in section 4.

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<sup>4</sup>This constraint arises from the time reparameterisation invariance of the action or, equivalently, from the 00 Einstein equation.

### Gauge Field

By adding a vector gauge field with Lagrangian

$$\mathcal{L} = -\sqrt{-g}e^{c\phi}F_{\mu\nu}F^{\mu\nu} \quad |_{y=-1}, \quad (5.33)$$

we obtain an effective theory described by the action

$$S = \int_{4d} [-\dot{a}_4^2 + a_4^2(\dot{\psi}^2 + \dot{\chi}^2) - a_4^4 e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)} F_{\mu\nu}F^{\mu\nu}]. \quad (5.34)$$

Then we have the constraint

$$\frac{\dot{a}_4^2}{a_4^2} = \dot{\psi}^2 + \dot{\chi}^2 + a_4^2 e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)} (2E^2 + 2B^2) \quad (5.35)$$

together with the equations of motion

$$\frac{\ddot{a}_4}{a_4} = -\dot{\psi}^2 - \dot{\chi}^2 \quad (5.36)$$

$$\ddot{\psi} + 2\frac{\dot{a}_4}{a_4}\dot{\psi} + \frac{1}{a_4^2}\frac{\partial}{\partial\psi}[e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi}(-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)}](-E^2 + B^2) = 0 \quad (5.37)$$

$$\ddot{\chi} + 2\frac{\dot{a}_4}{a_4}\dot{\chi} + \frac{1}{a_4^2}\frac{\partial}{\partial\chi}[e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi}(-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)}](-E^2 + B^2) = 0 \quad (5.38)$$

$$\partial^\mu[F_{\mu\nu}e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi}(-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)}] = 0. \quad (5.39)$$

The last equation, supplemented by the Bianchi identity

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \quad (5.40)$$

leads to

$$E = E_0 e^{2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{6\beta c/(3\beta^2+1)} \quad (5.41)$$

$$B_i = B_{i,0} \quad (5.42)$$

where  $E_0$  and  $B_{i,0}$  are constants. The equations of motion for  $\psi$  and  $\chi$  can then be rewritten as

$$\ddot{\psi} + 2\frac{\dot{a}_4}{a_4}\dot{\psi} + \frac{1}{a_4^2}V_{,\psi} = 0 \quad (5.43)$$

$$\ddot{\chi} + 2\frac{\dot{a}_4}{a_4}\dot{\chi} + \frac{1}{a_4^2}V_{,\chi} = 0 \quad (5.44)$$

with the effective potential

$$\begin{aligned}
V &= E_0^2 e^{2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{6\beta c/(3\beta^2+1)} \\
&+ B_0^2 e^{-2(c/\beta)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{-6\beta c/(3\beta^2+1)}.
\end{aligned} \tag{5.45}$$

Thus we can see that near  $\chi = 0$  the effective potential blows up and leads to a bounce of the negative-tension brane if we either have an electric field and

$$\beta c < 0, \tag{5.46}$$

or if we have a magnetic field and

$$\beta c > 0. \tag{5.47}$$

This is in agreement with the 5d description of section 4. Also, if we consider radiation, for which

$$\langle E^2 \rangle = \langle B^2 \rangle, \tag{5.48}$$

it is immediately apparent from equations (5.36)-(5.38) that it does not lead to a bounce. This is again consistent with the 5d results derived earlier.

### *Cosmological Constant*

We can repeat the above analysis in the case of a brane-localised cosmological constant  $\Lambda$ , also coupling to the scalar  $\phi$ . In that case the effective action receives an additional contribution of

$$-\sqrt{-g}e^{c\phi}2\Lambda \Big|_{y=-1} \tag{5.49}$$

$$= -a_4^4 e^{-2(c/\beta+2)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(-6\beta c+4)/(3\beta^2+1)} 2\Lambda. \tag{5.50}$$

Therefore, the effective potential is

$$V = \Lambda e^{-2(c/\beta+2)\sqrt{\frac{3\beta^2}{3\beta^2+1}}\psi} (-\sinh \sqrt{3\beta^2+1}\chi)^{(-6\beta c+4)/(3\beta^2+1)} \tag{5.51}$$

and as  $\chi \rightarrow 0$ , we get a bounce if

$$\beta c > 2/3 \quad (\Lambda > 0). \tag{5.52}$$

This is exactly the same requirement as that obtained from the 5d point of view for a positive cosmological constant.

However, the case of a negative cosmological constant cannot be reproduced within the 4d effective theory, as the effective potential is negative in that case.

## 6 Heterotic M-Theory Examples

Heterotic M-theory corresponds to the special case  $\beta = -1$ , with the scalar  $\phi$  parameterising the volume of the internal Calabi-Yau manifold [14]. It is in this theory that the colliding branes solution [1], which was briefly discussed in the introduction and which motivated the present work, was derived. The solution was described in a coordinate system in which the bulk is static and the branes are moving. The boundary conditions used correspond to requiring the brane scale factors and the Calabi-Yau volume to be non-zero and finite at the collision of the branes. This turns out to be equivalent to imposing the relationship [1]

$$\phi = 6a. \tag{6.1}$$

This condition relates the volume of the Calabi-Yau to the brane scale factors, while reducing the number of independent fields to two. This last feature enables one to derive a Birkhoff-like theorem <sup>5</sup>, which determines the bulk metric to be given by a one-parameter time-independent family of metrics (the parameter being the relative rapidity of the branes at the collision), with the branes moving in this background geometry according to their junction conditions. It is easy to see from the junction conditions (3.2)-(3.4) that we can keep the requirement that  $\phi = 6a$ , and thus the Birkhoff-like theorem mentioned above, only if

$$\mathcal{T}_0^0 = \frac{1}{2}\mathcal{T}_\phi. \tag{6.2}$$

Thus we can see that in general a very specific coupling  $C(\phi)$  to the Calabi-Yau volume scalar is required if we want the bulk spacetime to remain unaltered by the presence of brane-bound matter (the brane trajectories will of course be modified in any case).

For a brane-bound scalar, it is straightforward to see that the bulk geometry is unaltered only if the coupling is

$$C = e^\phi. \tag{6.3}$$

As shown in section 4, there will also be a bounce in this case, and the entire evolution can be described exactly, since the bulk spacetime is given by the solution described in [1]. From the moduli space point of view, we can note that the effective potential (5.31) is independent of  $\psi$  only for  $C = e^\phi$ , which coincides with the condition for the bulk geometry to be unaltered. This can be understood by the fact that, if the effective

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<sup>5</sup>For the case of general  $\beta$ , a similar Birkhoff-like theorem can be derived if one imposes  $\phi = -6\beta a$ . The discussion in the present section can be generalised in a straightforward, but unilluminating way to having arbitrary  $\beta$ .



potential is independent of  $\psi$ , the scalar field space trajectory reflects off the effective potential with the same final angle as the incident angle, in a smoothed-out version of a “brick wall” reflection at  $\chi = 0$ , and therefore the background trajectory is unchanged except for this symmetric rounding off of the trajectory near the bounce of the negative-tension brane. Thus, for scalar field matter, the 4d and 5d points of view are in perfect agreement. This can be traced back to the fact that we are simply extending the moduli space by one dimension, by adding an extra kinetic term, and therefore the moduli space description should remain a good approximation.

Note that the scalars arising from the dimensional reduction of the  $E_8$  gauge fields in heterotic M-theory do not couple to the Calabi-Yau volume, *i.e.* they have  $C = 1$  [15]. Scalars of this type also make the negative-tension brane bounce. However, the bulk geometry will be altered in this case, which is why it might be of interest to calculate the resulting deformed geometry.

For gauge fields, condition (6.2) shows that the bulk is unaltered only if

$$-(E^2 + B^2)C = (-E^2 + B^2)C_{,\phi}. \quad (6.4)$$

This can be satisfied either if we have an electric field only ( $B = 0$ ) with the coupling

$$C = e^\phi \quad (6.5)$$

or if we only have a magnetic field ( $E = 0$ ) and the coupling

$$C = e^{-\phi}. \quad (6.6)$$

However, in both cases, the effective potential (5.45) in the moduli space description is independent of  $\psi$  only if  $C = 1$ . While the moduli space approximation correctly predicts whether or not a bounce occurs, the detailed trajectory followed in this description is not perfectly symmetric about the bounce (when the coupling is such that the bulk remains unaltered), and hence not a perfect rendition of the 5d solution.

In fact, the  $E_8$  gauge fields in heterotic M-theory couple with  $C = e^\phi$  [15]. Their electric component therefore contributes to a bounce, while also leaving the bulk geometry unaltered, while their magnetic component rather contributes to a crunch (and a deformation of the bulk geometry).

Again by inspection of (6.2), it is easy to see that a brane-bound cosmological constant does not perturb the bulk geometry if its coupling is given by  $C = e^{-\phi}$ . In this case, we simply have a de-tuning of the brane tension. This de-tuning leads to a bounce if the

cosmological constant is positive, whereas it leads to a crunch if it is negative. Note that the moduli space description yields a potential (5.51) that is independent of  $\psi$  only when  $C = e^{2\phi}$ , which is in disagreement with the 5d description.

## 7 Conclusions

In a dynamical braneworld setting, the negative-tension boundary brane can encounter a zero of the harmonic function corresponding to the formation of a singularity. However, we have shown that this catastrophic encounter is avoided in the presence of a broad range of brane-bound matter types and couplings to the scalar field supporting the domain walls, which make the negative-tension brane bounce off the naked singularity<sup>6</sup>. This leads us to the rather surprising conclusion that negative-tension branes can stabilise braneworlds.

We have analysed the bounce of the negative-tension brane from two points of view: firstly, we have looked at the 5d equations of motion and junction conditions in the vicinity of the bounce. And secondly, we have analysed the analogous situation using the moduli space approximation. For scalar fields, the two descriptions are in perfect agreement. This is because adding a kinetic term is perfectly suited to the spirit of the moduli space approximation. For gauge fields and for a positive cosmological constant, the moduli space approach correctly reproduces the 5d results for the bounce. However, when the conditions are fulfilled for the 5d bulk to remain unaltered and we hence know that the 4d trajectory should be perfectly symmetric about the bounce, the 4d effective theory does not reproduce this behaviour. And in the case of a negative cosmological constant, the moduli space approach completely disagrees with the 5d results. It seems clear that in case of a disagreement, we should rather trust the 5d results. In fact, our results indicate that in the case of a brane-bound gauge field or a cosmological constant, the approximations used in deriving the moduli space action are not really valid. In these cases, there are non-flat directions in configuration space which are easily accessible to the system under study, and which are not described by the moduli space approximation. Thus, even though the moduli space description can give qualitatively correct results in describing the effects of a gauge field or a positive cosmological constant, the detailed quantitative analysis can be rather misleading, and one should revert to a 5d description.

The types of brane-bound matter that are naturally present in heterotic M-theory

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<sup>6</sup>Thus, we could say that we have a bang if no observer is there to hear it, but no sound in the presence of the right kind of observer!

are scalar fields that do not couple to the Calabi-Yau volume, and gauge fields with an  $e\phi$  coupling. What we found is that for this specific coupling, electric fields contribute towards a bounce, while radiation has no effect and magnetic fields rather contribute to a crunch. The scalars contribute towards a bounce, and probably represent the best candidates for stabilising the heterotic M-theory braneworld.

Finally, we would like to point out that it seems likely that additional brane-bound matter will be produced by quantum effects at the bounce of the negative-tension brane, and it would be interesting to determine the properties of these new contributions.

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